Recall from last lecture:

<u>Lemma 2</u> $\phi: X \dashrightarrow X'$ birational map of surfaces ~~[illegible]~~ s.t. $\phi^{-1}$ undefined at $p' \in X'$.

$\Rightarrow \exists\, C \subset X$ s.t. $\phi(C) = p'$

<u>Prop</u> $\pi: X' \to X$ birational morphism of surfaces s.t. $\pi^{-1}$ undefined at $p \in X$

$\Rightarrow \exists$ comm. diag of bir. morph.

$$\begin{tikzcd} X' \arrow[r, "\tilde{\pi}"] \arrow[dr, "\pi"'] & Bl_p X = \tilde{X} \arrow[d, "\beta"] \\ & X \end{tikzcd}$$

<u>Proof</u> Suppose $\tilde{\pi} = \beta^{-1} \circ \pi$ undefined at $p' \in X'$

Lemma 2 $\Rightarrow \exists\, \tilde{E} \subset \tilde{X}$ s.t. $\tilde{\pi}^{-1}(\tilde{E}) = p'$, $\beta(\tilde{E}) = p$

$\Rightarrow \tilde{E} = E$

Lemma 2 $\Rightarrow \exists\, C' \subset X'$ s.t. $\pi(C') = p$.

$Z_{\tilde{\pi}^{-1}}$ zero-dim $\Rightarrow \exists\, e \in E$ (wlog $e = \infty$) s.t. $\tilde{\pi}^{-1}$ defined at $e$.

$\Rightarrow$ have inclusions of subrings in $K = K(X) = K(X') = K(\tilde{X})$

$$\mathcal{O}_{X,p} \subseteq \mathcal{O}_{X',p'} \subseteq \mathcal{O}_{\tilde{X},e}$$

$x, y$ local param. $\quad f' \in m_{X',p'}$ local eqn for $C'$ $\quad x, \tilde{y} = \frac{y}{x}$ local parameters.

$\tilde{\pi}^{-1}(E) = p' \Rightarrow m_{X',p'} \subseteq (x)$

$\pi(C') = p \Rightarrow m_{X,p} \subseteq (f') \Rightarrow x = af'$, $y = bf'$ for $a, b \in \mathcal{O}_{X',p'}$

$x$ local parameter @ $e \Rightarrow x$ local parameter at $p'$

$\Rightarrow a \in \mathcal{O}^\times_{X',p'}$ a unit.

$bf' = y = x \cdot \frac{y}{x} = \frac{y}{x} \cdot a \cdot f' \Rightarrow \frac{y}{x} \cdot a = b$

$\Rightarrow \frac{y}{x} = \frac{b}{a} \in m_{\tilde{X},e} \cap \mathcal{O}_{X',p'} = m_{X',p'} \subseteq (x)$ ↯

<u>Theorem</u> $\pi: X' \to X$ birational morphism of surfaces

$\Rightarrow \exists$ sequence of blow ups at points

$$X_n \xrightarrow[\beta_n]{} X_{n-1} \to \cdots \xrightarrow[\beta_2]{} X_1 \xrightarrow[\beta_1]{} X$$

and iso $\eta: X' \to X_n$ s.t. $\pi = \beta_1 \circ \cdots \circ \beta_n \circ \eta$.

<u>Proof</u> if $\pi$ iso, done. else $\exists p_1 \in X$ s.t. $\pi^{-1}$ undef. @ $p_1$

Prep. $\Rightarrow$ $\pi$ factors through blow up at $p_1$.

$$\pi = \beta_1 \circ \pi_1.$$

repeat with $\pi_1$.

$0 \leq n(\pi_i) := \#\{\text{irred curves } C \subseteq X' \text{ contracted by } \pi_i\}$.

$C_i := \pi_i^{-1}(E_i) \Rightarrow \pi_i(C_i) = E_i$

$\Rightarrow$ $C_i$ contracted by $\pi_{i-1}$ but not $\pi_i$

$\Rightarrow$ $n(\pi_i) < n(\pi_{i-1})$

$\Rightarrow$ process terminates. $\square$

<u>Corollary</u> $\phi: X \dashrightarrow X'$ birational map of surfaces

$\Rightarrow \exists$ commutative diagram.
$$\begin{tikzcd} & \tilde{X} \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] & \\ X \arrow[rr, dashed, "\phi"] & & X' \end{tikzcd}$$

s.t. $\pi, \pi'$ are composition of blow ups at pts and isos $\square$

Q: given $X$ when can we contract a curve $C \subset X$?

<u>Prop.</u> $X$ surface, $C \subseteq X$ ns. curve, $C \cong \mathbb{P}^1$, $C^2 < 0$
$\Rightarrow \exists$ birational morphism $\pi : X \to X'$ s.t. $\pi(C) = pt$ and $X \setminus C \xrightarrow{\pi} X' - \pi(C)$ iso. $X'$ proj. var.

<u>Proof</u> suffices to find linear system $|D|$ glob. gen. s.t. 1) separates points and tangent vectors on $X \setminus C$
2) $D|_C$ degree zero divisor on $\mathbb{P}^1$
$\Rightarrow$ linear system contracts $C$

let $H$ be very ample on $X$ giving embedding $X \hookrightarrow \mathbb{P}^N = \mathbb{P}(H^0(\mathcal{O}_X(H))$
1) can be achieved by only modifying $H$ at $C$:
Consider divisors $iH + jC$ (rmk: $iH + jC|_{X \setminus C} = iH|_{X \setminus C}$ ⊛)
want $iH + jC$ to be globally gen. consider s.e.s.

$$0 \to \mathcal{O}_X(iH + (j-1)C) \to \mathcal{O}_X(iH + jC) \to \underbrace{\mathcal{O}_X(iH + jC)|_C}_{\cong \mathcal{O}_{\mathbb{P}^1}(i(H.C) + jC^2)} \to 0$$

l.e.s. in cohomology.

⊛⊛ $0 \to H^0(\mathcal{O}_X(iH + (j-1)C)) \to H^0(\mathcal{O}_X(iH + jC)) \to H^0(\mathcal{O}_X(iH + jC)|_C) \to$
$\to H^1(\mathcal{O}_X(iH + (j-1)C)) \to H^1(\mathcal{O}_X(iH + jC)) \to \underbrace{H^1(\mathcal{O}_X(iH + jC)|_C)}_{=0 \text{ if } i(H.C) + j \cdot C^2 \geq 0} \to$

$\Rightarrow$ for fixed $i$, $j = 1, \dots, \frac{-i(H.C)}{C^2}$ use negative self int.
induction on $j$ $\Rightarrow H^1(\mathcal{O}_X(iH + jC)) = 0$
$i(H.C) + j \cdot C^2) \geq 0 \Rightarrow \mathcal{O}_X(iH + jC)|_C$ glob. gen.
⊛ + ⊛⊛ + Nakayama $\Rightarrow \mathcal{O}_X(iH + jC)$ glob gen. $\Rightarrow$

(*) $\Rightarrow$ 1) for $\mathcal{O}_X(iH+jC)$ if $i(H.C)+j\,C^2 \geq 0$

check: $\deg(\mathcal{O}_X(iH+jC))|_C = i(H.C)+j\,C^2$

$\Rightarrow \deg(\mathcal{O}_X(-C^2H+(H.C)C)) = 0$

$\Rightarrow$ 2) $\square$

Q: When is $X'$ nonsingular?

**Theorem** (Castelnuovo's Criterion)

$X$ surface, $C \subseteq X$ s.t. $C \cong \mathbb{P}^1$, $C^2 = -1$. "exceptional curve"

$\Rightarrow \exists$ surface $X'$, $p' \in X'$ s.t.
$X = Bl_{p'} X'$ and $C = E_{p'} X'$

**Proof idea:**

1) Construct $\pi : X \to X'$ as in previous proposition.

2) Apply "Theorem on Formal functions" to show that $X'$ is nonsingular.
The condition $C^2 = -1$ is used to show $\mathcal{I}_C / \mathcal{I}_C^2 = \mathcal{O}_{\mathbb{P}^1}(1)$ so that

$$\widehat{\mathcal{O}}_{X', p'} = \varprojlim k[x,y] / (x,y)^n$$

$H^0(\mathcal{O}_{\mathbb{P}^1}(1)) = \mathbb{C}x \oplus \mathbb{C}y$. $= k[x,y]$

3) By Prop / factorization theorem
$\pi$ must be a blow up $\square$.

**Facts** (follow from theorem on formal functions)

- $\beta : \tilde{X} = Bl_p X \to X$

$\Rightarrow$ • $\beta_* \mathcal{O}_{\tilde{X}} = \mathcal{O}_X$ and $R^i \beta_* \mathcal{O}_{\tilde{X}} = 0$ for $i > 0$
in particular $H^i(\mathcal{O}_{\tilde{X}}) = H^i(\mathcal{O}_X)$

---

Q: How to classify surfaces?
Hard b/c can blow up to get new surface.

<u>Def</u> A surface $X$ is <u>minimal</u> if every birational morphism $X \to X'$ to a surface is an isomorphism.

Castelnuovo $\Rightarrow$ $X$ minimal, then $X$ has no exceptional curves

<u>Prop</u> Every surface dominates a minimal surface (i.e. $X$ surface $\exists\, X \to X_0$ bir mor s.t. $X_0$ minimal).

<u>Proof</u> if $X$ is not minimal, by factorization theorem, there exists a blow down $\beta : X = Bl_{p_1} X' \to X'$

$\Rightarrow \operatorname{Pic}(X) = \operatorname{Pic}(X') \oplus \mathbb{Z}$

would like to use rank of $\operatorname{Pic}(X)$ as invariant that drops after each blow down, but don't know that $\operatorname{Pic}(X)$ finitely generated.

have $\operatorname{Pic}(X) \cong H^1(\mathcal{O}_X^*) \longrightarrow H^1(\Omega_X)$

$$f \longmapsto f^{-1} df$$

$\leadsto$ try to use fin. dim. v.s. $H^1(\Omega_X)$

$R^0\beta_* \mathcal{O}_{X'} = \mathcal{O}_X$ & proj. formula $R^i\beta_*(\mathcal{O}_X \otimes \beta^*\Omega_{X'}) = R^i\beta_*\mathcal{O}_{X'} \otimes \Omega_{X'}$

$\Rightarrow H^i(X', \Omega_{X'}) = H^i(X, \pi^*\Omega_{X'})$

s.e.s. $0 \to \pi^*\Omega_{X'} \to \Omega_X \to \Omega_{X/X'} \to 0$

$$\Omega_{X/X'} \cong \Omega_E = \mathcal{O}_{\mathbb{P}^1}(-2)$$

l.e.s. $0 \to H^0(X', \Omega_{X'}) \xrightarrow{\cong} H^0(X, \Omega_X) \to H^0(\Omega_E) = 0$

$\hookrightarrow H^1(X', \Omega_{X'}) \to H^1(X, \Omega_X) \to H^1(\Omega_E) \cong k$

$\hookrightarrow H^2(X', \Omega_{X'}) \xrightarrow{\cong} H^2(X, \Omega_X) \to 0$

$H^2(X', \Omega_{X'}) \cong H^0(X', \Omega_{X'})^*$, $\quad H^2(X, \Omega_X) \cong H^0(X, \Omega_X)^*$

$\Rightarrow H^1(X^*, \Omega_X) = H^1(X', \Omega_{X'}) \oplus k$.

$\square$

<u>Proof #2:</u> $\langle \, , \rangle : H^1(\Omega_X) \times H^1(\Omega_X) \longrightarrow k$

Serre duality, perfect pairing.

exercise: compatible with intersection pairing.

$c : \mathrm{Pic}(X) \to H^1(\Omega_X)$

$$\langle c(D) . c(D') \rangle = (D . D') \cdot 1 \in k.$$

~~blow downs $X \to X'$ come from exceptional curves $\mathrm{Exc}(X) \subseteq \mathrm{Pic}(X)$~~ Kronecker $\delta$

~~$E, E' \in \mathrm{Exc}(X) \Rightarrow \langle c(E), c(E') \rangle = -\delta_{EE'}$~~

~~$\Rightarrow c(\mathrm{Exc}(X)) \subseteq H^1(\Omega_X)$ linearly indep.~~

$$X = X_0 \xrightarrow{\beta_1} X_1 \to \cdots \to X_n$$

sequence of $n$ blow downs. $E_i' \subseteq X_i$ except. divisor.

$E_1, \dots, E_n \in \mathrm{Pic}(X)$

$E_i := (\beta_i \circ \dots \circ \beta_1)^* E_i'$ total transform.

$\Rightarrow \langle c(E_i), c(E_j) \rangle = (E_i . E_j) \cdot 1 = -\delta_{ij}$

$\Rightarrow c(E_1), \dots, c(E_n) \subseteq H^1(\Omega_X)$ linearly independent.

$\Rightarrow n \leq h^1(\Omega_X)$ $\square$

<u>Remark</u> 1) minimal models need not be unique.

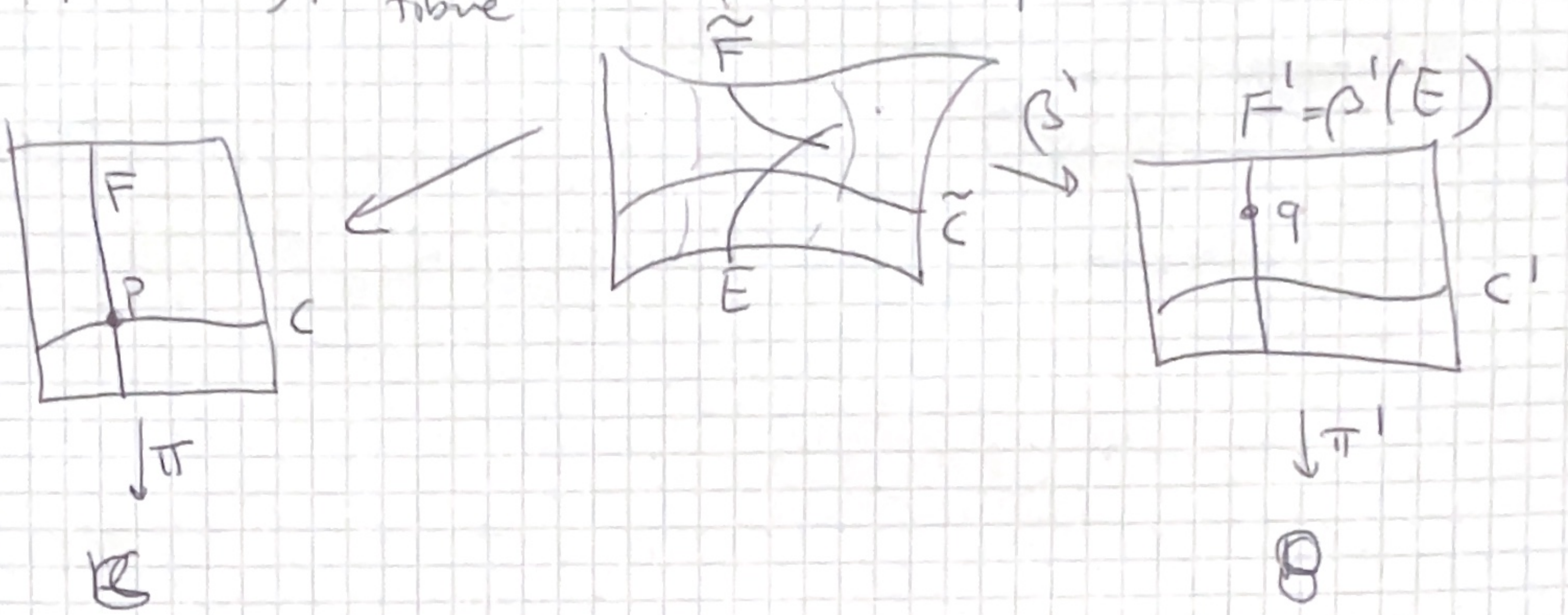
$$\begin{tikzcd} \mathbb{P}^2 \text{ is minimal.} & & \mathbb{P}^1 \times \mathbb{P}^1 \text{ is minimal.} \\ & \mathrm{Bl}_{P_1, P_2} \mathbb{P}^2 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

2) there exist ~~curves~~ surfaces with infinitely many exceptional curves.

# Elementary transformations

$\pi : X \to B$ geometrically ruled surface.

$p \in X$, $p \in F \subset X$. $\beta : \tilde X = Bl_p X \longrightarrow X$
fibre




$$F^2 = 0 \Rightarrow \underbrace{\tilde F^2}_{=-1} = (\beta^* F - E)^2 = F^2 - 2\overbrace{(\beta^* F . E)}^{=0} + E^2$$

$\Rightarrow$ can contract $\tilde F$ $\quad \beta' : \tilde X \longrightarrow X'$ $\quad q := \beta'(\tilde F)$

$$F' := \beta'(E), \quad (F')^2 = (\underbrace{\tilde F'}_{=\tilde E} + \tilde F)^2$$
$$= E^2 + 2\underbrace{(\tilde F . E)}_{=1} + \tilde F^2 = 0$$

$\pi' := \pi \circ \beta \circ (\beta')^{-1}$ is a morphism.
~~(at most not defined at q)~~ (maps from surfaces to curves always defined everywhere).

and get a section
$\sigma' = \beta' \circ \beta^{-1} \circ \sigma$

$\Rightarrow$ $\pi' : X' \longrightarrow B$ ruled surface called elementary transformation of $X \longrightarrow B$

---

Birational Classification of surfaces reduces to classifying minimal models.